Section 1.2 Example 2 with general right hand sides: Ax=y.

Standard forward pivoting, unscaled pivots.

Consistency condition derived and used halfway: some y are slaved.

The smaller reduced augmented matrix pivoted backward, again with unscaled pivots.

Pivots scaled at the end.

Full solution at the end. Some x are killed.

Killing the free x and forgetting the slaved y.

Linearly independent v's that span N(A).

Linearly independent w's that span R(A).

Let's see how. Introduce augmented matrix with general y's, one in every column:

$$\begin{bmatrix} 0 & -3 & -6 & 4 & 9 & y_1 & 0 & 0 & 0 \\ -1 & -2 & -1 & 3 & 1 & 0 & y_2 & 0 & 0 \\ -2 & -3 & 0 & 3 & -1 & 0 & 0 & y_3 & 0 \\ 1 & 4 & 5 & -9 & -7 & 0 & 0 & 0 & y_4 \end{bmatrix} \quad (1)$$

Extract the rows
and build the equations:

$$\begin{gathered} -3x_2 - 6x_3 + 4x_4 + 9x_5 = y_1 \\ -x_1 - 2x_2 - x_3 + 3x_4 + x_5 = y_2 \\ -2x_1 - 3x_2 + 3x_4 - x_5 = y_3 \\ x_1 + 4x_2 + 5x_3 - 9x_4 - 7x_5 = y_4 \end{gathered} \quad (2)$$

Note that

$$A = \begin{bmatrix} 0 & -3 & -6 & 4 & 9 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 1 & 4 & 5 & -9 & -7 \end{bmatrix} \quad (3)$$

The augmented matrix:

$$\begin{bmatrix} 0 & -3 & -6 & 4 & 9 & y_1 & 0 & 0 & 0 \\ -1 & -2 & -1 & 3 & 1 & 0 & y_2 & 0 & 0 \\ -2 & -3 & 0 & 3 & -1 & 0 & 0 & y_3 & 0 \\ 1 & 4 & 5 & -9 & -7 & 0 & 0 & 0 & y_4 \end{bmatrix} \quad (4)$$

We do row operations.
Every row operation leads to a new augmented matrix coding an equivalent new system of linear equations.

Forward pivot phase.

Choose a pivot row with the pivot in position 1 and put it on top:

```
> R[0]:=R[1]:R[1]:=R[2]:R[2]:=R[0]:
```

$$\begin{bmatrix} -1 & -2 & -1 & 3 & 1 & 0 & y_2 & 0 & 0 \\ 0 & -3 & -6 & 4 & 9 & y_1 & 0 & 0 & 0 \\ -2 & -3 & 0 & 3 & -1 & 0 & 0 & y_3 & 0 \\ 1 & 4 & 5 & -9 & -7 & 0 & 0 & 0 & y_4 \end{bmatrix} \quad (5)$$

Don't scale the pivot yet and let it kill below in its (first) pivot column:

```
> R[2]:=R[2]-R[2][1]*R[1]/R[1][1]:
> R[3]:=R[3]-R[3][1]*R[1]/R[1][1]:
> R[4]:=R[4]-R[4][1]*R[1]/R[1][1]:
```

$$\begin{bmatrix} -1 & -2 & -1 & 3 & 1 & 0 & y_2 & 0 & 0 \\ 0 & -3 & -6 & 4 & 9 & y_1 & 0 & 0 & 0 \\ 0 & 1 & 2 & -3 & -3 & 0 & -2\,y_2 & y_3 & 0 \\ 0 & 2 & 4 & -6 & -6 & 0 & y_2 & 0 & y_4 \end{bmatrix} \quad (6)$$

Choose the next pivot in column 2 and row 2 and let it kill below in its (second) pivot column:

```
> R[3]:=R[3]-R[3][2]*R[2]/R[2][2]:
> R[4]:=R[4]-R[4][2]*R[2]/R[2][2]:
```

$$\begin{bmatrix} -1 & -2 & -1 & 3 & 1 & 0 & y_2 & 0 & 0 \\ 0 & -3 & -6 & 4 & 9 & y_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{5}{3} & 0 & \frac{y_1}{3} & -2\,y_2 & y_3 & 0 \\ 0 & 0 & 0 & -\frac{10}{3} & 0 & \frac{2\,y_1}{3} & y_2 & 0 & y_4 \end{bmatrix} \quad (7)$$

The next pivot is in column 4 and row 3. Let it kill below in its (third) pivot column:

```
> R[4]:=R[4]-R[4][4]*R[3]/R[3][4]:
```

$$\begin{bmatrix} -1 & -2 & -1 & 3 & 1 & 0 & y_2 & 0 & 0 \\ 0 & -3 & -6 & 4 & 9 & y_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{5}{3} & 0 & \frac{y_1}{3} & -2\,y_2 & y_3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 5\,y_2 & -2\,y_3 & y_4 \end{bmatrix} \tag{8}$$

Build the new equations:

$$\begin{gathered} -x_1 - 2\,x_2 - x_3 + 3\,x_4 + x_5 = y_2 \\ -3\,x_2 - 6\,x_3 + 4\,x_4 + 9\,x_5 = y_1 \\ -\frac{5\,x_4}{3} = \frac{y_1}{3} - 2\,y_2 + y_3 \\ 0 = 5\,y_2 - 2\,y_3 + y_4 \end{gathered} \tag{9}$$

The consistency condition is:

$$y_4 := -5\,y_2 + 2\,y_3 \tag{10}$$

It makes the last equation 0=0 and lets us forget about the last y. It is slaved to the first 3 y's.

Skip last row and last column of the augmented matrix:

$$Aug := \begin{bmatrix} -1 & -2 & -1 & 3 & 1 & 0 & y_2 & 0 \\ 0 & -3 & -6 & 4 & 9 & y_1 & 0 & 0 \\ 0 & 0 & 0 & -\frac{5}{3} & 0 & \frac{y_1}{3} & -2\,y_2 & y_3 \end{bmatrix} \tag{11}$$

Build the remaining equations:

$$\begin{gathered} -x_1 - 2\,x_2 - x_3 + 3\,x_4 + x_5 = y_2 \\ -3\,x_2 - 6\,x_3 + 4\,x_4 + 9\,x_5 = y_1 \\ -\frac{5\,x_4}{3} = \frac{y_1}{3} - 2\,y_2 + y_3 \end{gathered} \tag{12}$$

Backward pivot phase for the smaller augmented matrix:

```
> R[2]:=R[2]-R[2][4]*R[3]/R[3][4]:R[1]:=R[1]-R[1][4]*R[3]/R[3][4]:
```

$$\begin{bmatrix} -1 & -2 & -1 & 0 & 1 & \frac{3\,y_1}{5} & -\frac{13\,y_2}{5} & \frac{9\,y_3}{5} \\ 0 & -3 & -6 & 0 & 9 & \frac{9\,y_1}{5} & -\frac{24\,y_2}{5} & \frac{12\,y_3}{5} \\ 0 & 0 & 0 & -\frac{5}{3} & 0 & \frac{y_1}{3} & -2\,y_2 & y_3 \end{bmatrix} \tag{13}$$

```
> R[1]:=R[1]-R[1][2]*R[2]/R[2][2]:
```

$$\begin{bmatrix} -1 & 0 & 3 & 0 & -5 & -\frac{3y_1}{5} & \frac{3y_2}{5} & \frac{y_3}{5} \\ 0 & -3 & -6 & 0 & 9 & \frac{9y_1}{5} & -\frac{24y_2}{5} & \frac{12y_3}{5} \\ 0 & 0 & 0 & -\frac{5}{3} & 0 & \frac{y_1}{3} & -2y_2 & y_3 \end{bmatrix} \quad (14)$$

Scale the pivots:

```
> R[1]:=R[1]/R[1][1]:
> R[2]:=R[2]/R[2][2]:
> R[3]:=R[3]/R[3][4]:
```

$$\begin{bmatrix} 1 & 0 & -3 & 0 & 5 & \frac{3y_1}{5} & -\frac{3y_2}{5} & -\frac{y_3}{5} \\ 0 & 1 & 2 & 0 & -3 & -\frac{3y_1}{5} & \frac{8y_2}{5} & -\frac{4y_3}{5} \\ 0 & 0 & 0 & 1 & 0 & -\frac{y_1}{5} & \frac{6y_2}{5} & -\frac{3y_3}{5} \end{bmatrix} \quad (15)$$

Do you see which variables are free? We will kill them later.

Extract the equations:

$$x_1 - 3x_3 + 5x_5 = \frac{3y_1}{5} - \frac{3y_2}{5} - \frac{y_3}{5}$$
$$x_2 + 2x_3 - 3x_5 = -\frac{3y_1}{5} + \frac{8y_2}{5} - \frac{4y_3}{5}$$
$$x_4 = -\frac{y_1}{5} + \frac{6y_2}{5} - \frac{3y_3}{5} \quad (16)$$

Solve backwards with free variables replaced by t's:

$$x_5 := t_5 \quad (17)$$

$$x_4 := -\frac{y_1}{5} + \frac{6y_2}{5} - \frac{3y_3}{5} \quad (18)$$

$$x_3 := t_3 \quad (19)$$

$$x_2 := -\frac{3y_1}{5} + \frac{8y_2}{5} - \frac{4y_3}{5} - 2t_3 + 3t_5 \quad (20)$$

$$x_1 := \frac{3y_1}{5} - \frac{3y_2}{5} - \frac{y_3}{5} + 3t_3 - 5t_5 \quad (21)$$

In vector form the solution is:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} \frac{3y_1}{5} - \frac{3y_2}{5} - \frac{y_3}{5} + 3t_3 - 5t_5 \\ -\frac{3y_1}{5} + \frac{8y_2}{5} - \frac{4y_3}{5} - 2t_3 + 3t_5 \\ t_3 \\ -\frac{y_1}{5} + \frac{6y_2}{5} - \frac{3y_3}{5} \\ t_5 \end{bmatrix} \quad \textbf{(22)}$$

Introduce the vectors

$$v_3 := \begin{bmatrix} 3 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$v_5 := \begin{bmatrix} -5 \\ 3 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$p := \begin{bmatrix} \frac{3y_1}{5} - \frac{3y_2}{5} - \frac{y_3}{5} \\ -\frac{3y_1}{5} + \frac{8y_2}{5} - \frac{4y_3}{5} \\ 0 \\ -\frac{y_1}{5} + \frac{6y_2}{5} - \frac{3y_3}{5} \\ 0 \end{bmatrix} \quad \textbf{(23)}$$

Then the solution is given by

$$x = p + t_3 v_3 + t_5 v_5 \quad \textbf{(24)}$$

We have seen why these v's are linearly independent. Note that Ax=0 has the same solution with p=0.

Solving Ax=0 we have also seen that the columns of A are linearly independent only if there are no free variables, that is, if x=0 is the only solution of Ax=0.

Recalling that the (slaving) consistency condition was

$$y_4 = -5y_2 + 2y_3 \quad \textbf{(25)}$$

we note that p is a linear combination of

$$\begin{bmatrix} \frac{3}{5} \\ -\frac{3}{5} \\ 0 \\ -\frac{1}{5} \\ 0 \end{bmatrix}, \begin{bmatrix} -\frac{3}{5} \\ \frac{8}{5} \\ 0 \\ \frac{6}{5} \\ 0 \end{bmatrix}, \begin{bmatrix} -\frac{1}{5} \\ -\frac{4}{5} \\ 0 \\ -\frac{3}{5} \\ 0 \end{bmatrix} \quad (26)$$

with y-coefficients. These 3 vectors are linearly independent.

Why's that? Well, just run the pivoting scheme again with the free variables set equal to zero.

Let's go.

Killing the free by

$$x_3 := 0$$
$$x_5 := 0 \quad (27)$$

the equations reduce to

$$-3\,x_2 + 4\,x_4 = y_1 \quad (28)$$
$$-x_1 - 2\,x_2 + 3\,x_4 = y_2$$
$$-2\,x_1 - 3\,x_2 + 3\,x_4 = y_3$$

The augmented matrix is then

$$\begin{bmatrix} 0 & -3 & 4 & y_1 \\ -1 & -2 & 3 & y_2 \\ -2 & -3 & 3 & y_3 \end{bmatrix} \quad (29)$$

Do the pivoting

```
> R[0]:=R[1]:R[1]:=R[2]:R[2]:=R[0]:
> R[2]:=R[2]-R[2][1]*R[1]/R[1][1]:
> R[3]:=R[3]-R[3][1]*R[1]/R[1][1]:
> R[4]:=R[4]-R[4][1]*R[1]/R[1][1]:
> R[3]:=R[3]-R[3][2]*R[2]/R[2][2]:
> R[4]:=R[4]-R[4][2]*R[2]/R[2][2]:
```

$$\begin{bmatrix} -1 & -2 & 3 & y_2 \\ 0 & -3 & 4 & y_1 \\ 0 & 0 & -\frac{5}{3} & \frac{y_1}{3} - 2\,y_2 + y_3 \end{bmatrix} \quad (30)$$

```
> R[2]:=R[2]-R[2][3]*R[3]/R[3][3]:
  R[1]:=R[1]-R[1][3]*R[3]/R[3][3]:
```

$$\begin{bmatrix} -1 & -2 & 0 & -\frac{13 y_2}{5} + \frac{3 y_1}{5} + \frac{9 y_3}{5} \\ 0 & -3 & 0 & \frac{9 y_1}{5} - \frac{24 y_2}{5} + \frac{12 y_3}{5} \\ 0 & 0 & -\frac{5}{3} & \frac{y_1}{3} - 2 y_2 + y_3 \end{bmatrix} \quad (31)$$

```
> R[1]:=R[1]-R[1][2]*R[2]/R[2][2]:
```

$$\begin{bmatrix} -1 & 0 & 0 & \frac{3 y_2}{5} - \frac{3 y_1}{5} + \frac{y_3}{5} \\ 0 & -3 & 0 & \frac{9 y_1}{5} - \frac{24 y_2}{5} + \frac{12 y_3}{5} \\ 0 & 0 & -\frac{5}{3} & \frac{y_1}{3} - 2 y_2 + y_3 \end{bmatrix} \quad (32)$$

```
> R[1]:=R[1]/R[1][1]:
> R[2]:=R[2]/R[2][2]:
> R[3]:=R[3]/R[3][3]:
```

Now we have that

$$\begin{bmatrix} 1 & 0 & 0 & \frac{3 y_1}{5} - \frac{3 y_2}{5} - \frac{y_3}{5} \\ 0 & 1 & 0 & -\frac{3 y_1}{5} + \frac{8 y_2}{5} - \frac{4 y_3}{5} \\ 0 & 0 & 1 & -\frac{y_1}{5} + \frac{6 y_2}{5} - \frac{3 y_3}{5} \end{bmatrix} \quad (33)$$

encodes the equations

$$x_1 = \frac{3}{5} y_1 - \frac{3}{5} y_2 - \frac{1}{5} y_3 \quad (34)$$
$$x_2 = -\frac{3}{5} y_1 + \frac{8}{5} y_2 - \frac{4}{5} y_3$$
$$x_4 = -\frac{1}{5} y_1 + \frac{6}{5} y_2 - \frac{3}{5} y_3$$

but these are the solution formula's for the reduced system

$$-3 x_2 + 4 x_4 = y_1 \quad (35)$$

$$-x_1 - 2 x_2 + 3 x_4 = y_2 \quad (36)$$

$$-2 x_1 - 3 x_2 + 3 x_4 = y_3 \quad (37)$$

Killing the free and forgetting the slaved we're back to a square matrix and its inverse. Which matrices?

In particular we see that the range R(A) is given by all vectors of the form

$$y = \begin{bmatrix} \frac{3y_2}{5} - \frac{3y_1}{5} + \frac{y_3}{5} \\ \frac{9y_1}{5} - \frac{24y_2}{5} + \frac{12y_3}{5} \\ \frac{y_1}{3} - 2y_2 + y_3 \\ -5y_2 + 2y_3 \end{bmatrix} \tag{38}$$

Each vector y in R(A) is thereby a unique linear combination

$$y = s_1 w_1 + s_2 w_2 + s_3 w_3 \tag{39}$$

of the linearly independent vectors

$$w_1 = \begin{bmatrix} -\frac{3}{5} \\ \frac{9}{5} \\ \frac{1}{3} \\ 0 \end{bmatrix}, w_2 = \begin{bmatrix} \frac{3}{5} \\ -\frac{24}{5} \\ -2 \\ -5 \end{bmatrix}, w_3 = \begin{bmatrix} \frac{1}{5} \\ \frac{12}{5} \\ 1 \\ 2 \end{bmatrix} \tag{40}$$

with parameters

$$y_1 = s_1, y_2 = s_2, y_3 = s_3 \tag{41}$$

just like the null space N(A) was given by

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 3t_3 - 5t_5 \\ -2t_3 + 3t_5 \\ t_3 \\ 0 \\ t_5 \end{bmatrix} \tag{42}$$

or

$$x = t_3 v_3 + t_5 v_5 \tag{43}$$

with parameters the free variables

$$x_3 = t_3, x_5 = t_5 \tag{44}$$